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# Designing parking facilities for autonomous vehicles

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ABSTRACT

Autonomous vehicles will have a major impact on parking facility designs in the future. Compared to regular car-parks that have only two rows of vehicles in each island, future car-parks (for autonomous vehicles) can have multiple rows of vehicles stacked behind each other. Although this multi-row layout reduces parking space, it can cause blockage if a certain vehicle is barricaded by other vehicles and cannot leave the facility. To release barricaded vehicles, the car-park operator has to relocate some of the vehicles to create a clear pathway for the blocked vehicle to exit. The extent of vehicle relocation depends on the layout design of the car-park. To find the optimal car-park layout with minimum relocations, we present a mixed-integer non-linear program that treats each island in the car-park as a queuing system. We solve the problem using Benders decomposition for an exact answer and we present a heuristic algorithm to find a reasonable upper-bound of the mathematical model. We show that autonomous vehicle car-parks can decrease the need for parking space by an average of 62% and a maximum of 87%. This revitalization of space that was previously used for parking can be socially beneficial if car-parks are converted into commercial and residential land-uses.



## 1. Introduction

Parking is an important part of transportation planning because a typical vehicle spends 95% of its lifetime sitting in a parking spot (Mitchell, 2015). The increasing need to store vehicles has transformed a lot of valuable real-estate into parking garages in many countries. In the United States, approximately 6500 square miles of land is devoted to parking which is larger than the entire state of Connecticut (Chester et al., 2011; Thompson, 2016). Allocating valuable land to parking increases renting costs and parking acquisition costs in major downtown cores. One example is Hong Kong where the average cost of one parking space is as high as 180 thousand USD (South China Morning Post, 2015). Realizing the high social cost of parking provision, *Autonomous Vehicle* (AV) industry leaders are rethinking how to reduce the parking footprint by converting traditional parking lots into automated parking facilities that can store more AVs (compared to regular vehicles) in smaller areas. In this study, we investigate the optimal design and management of such facilities.

AVs can reduce the parking footprint in several ways. As vehicles become driver-less, the passengers no longer need to be physically present in car-parks. Driver-less AVs drop off their passengers at the parking entrance (or at a designated drop-off zone) and head to a spot chosen by the car-park operator. In this automated parking system, the average space per vehicle

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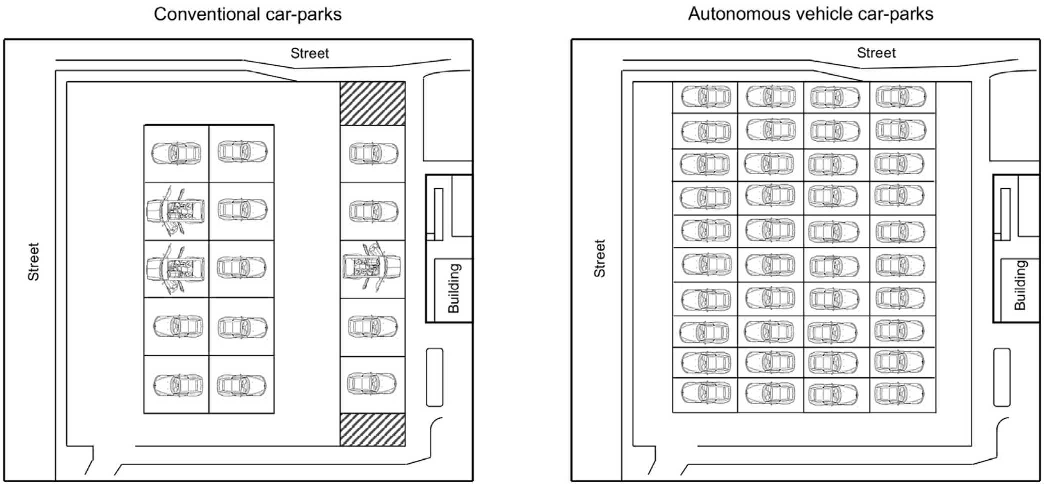


**Fig. 1.** (a) Conventional parking design, (b) autonomous vehicle parking design.

is estimated to decrease by 2 square meters per vehicle because the driving lanes become narrower, elevators and staircases become obsolete, and the required room for opening a vehicle's doors becomes unnecessary (see Fig. 1) (TechWorld, 2016).

Motivated by the benefits of AV parking and its impact on revitalizing valuable real-estate, auto makers are collaborating with cities to create the first generation of AV parking facilities. Audi's *Urban Futures Initiative* is among the programs that are implementing a pilot to measure the impact of AV parking on land restoration. The pilot is estimated to save up to 60% in parking space by 2030 which is equivalent to $100 million USD in the district of *Assembly Row* which is the focus of the project (DesignBoom, 2015). Tesla, another leader in AV technology, is also improving parking by offering an auto-pilot system called *"Smart Summon"* which allows the vehicle to navigate complex environments and parking spaces whenever summoned by its owner. Such auto-parking systems in AVs will pave the way for the next generation of AV parking facilities with improved space efficiency.

An essential strategy to increase car-park space efficiency (in addition to removal of elevators, etc.) is to stack the AVs in several rows, one behind the other as shown in Fig. 1b. While this type of layout reduces parking space, it can cause blockage if a certain vehicle is barricaded by other vehicles and cannot leave the facility. To release barricaded vehicles, the car-park operator has to relocate some of the vehicles to create a clear pathway for the blocked vehicle to exit. The extent of vehicle relocation depends on the layout of the car-park which should ideally be designed so that parking occupancy (i.e., number of vehicles in the car-park) is high and vehicle relocation is low. Of course, these autonomous facilities become more economically viable at higher AV penetration rates. However, parking facilities that service both vehicle types can be divided into two exclusive lots, one for the AVs and one for regular vehicles.

The layout of the car-park has a great impact on space efficiency. Existing layouts divide the car-park into a number of islands and roadways. The islands are used to store vehicles while the roadways separate the islands and allow vehicles to maneuver when searching for a desirable spot. To ensure that no vehicle gets blocked, islands hold no more than two rows of vehicles in conventional car-park designs (see Fig. 1a) which leads to waste of space. With AV technology, however, the islands can have more than two rows and the roadways can be narrower. An eminent research question that arises is: *How should we design AV parking facilities that minimize the relocations while satisfying a given demand?* To answer this question, we pursue the following objectives throughout this study:

- We present a model to find the optimal layout of a parking facility for AVs.
- We define a relocation strategy that ensures a smooth retrieval of any AV that is summoned by its user.
- We present exact and heuristic algorithms to find the optimal AV car-park layout.
- We find the maximum number of AVs that fit into a car-park with given dimensions.
- We quantify the potential parking space reduction when the car-park is exclusively designed for a given number of AVs.

The remainder of this paper is organized as follows: We present a background on AV models in Section 2, a model to find the optimal car-park layout in Section 3, two solution algorithms in Section 4, numerical experiments in Section 5, and the conclusions of the study in Section 6.

## 2. Background

With the rapid advancement in AV technologies, AVs are expected to enter the consumer market in the next decade (Fagnant and Kockelman, 2015). Already, AV technology leaders such as Google's *Waymo* have tested AVs over more than 3 million miles in several U.S. cities (Waymo, 2016). While full implementation of AVs is hindered by many practical challenges, recent studies are looking at ways of addressing these challenges and finding novel ways of exploiting the full potential of AVs. Current AV studies focus on traffic flow (Levin and Boyles, 2016; de Almeida Correia and van Arem, 2016; Mahmassani, 2016; Talebpour and Mahmassani, 2016; de Oliveira, 2017), safety (Katrakazas et al., 2015; Kalra and Paddock, 2016), intersection control (Le Vine et al., 2015; Yang and Monterola, 2016), emissions (Greenblatt and Saxena, 2015; Mersky and Samaras, 2016), and sharing AVs between multiple users (Fagnant and Kockelman, 2014; Chen et al., 2016a, 2016b; Krueger et al., 2016). AVs will also significantly change parking behavior by allowing the vehicles to self-park, a trait that the general public is very enthusiastic about. In a recent survey of 5600 people in 10 countries conducted by the World Economic Forum, 43.5% of the respondents reported that the biggest benefit of AV technology is its self-parking capability (Mitchell, 2015).

Parking behavior is argued to be influenced by AVs in several ways. Given that they can self-park, AVs no longer need to be in close proximity of their drivers. Instead, they can be dispatched to less congested parking lots that are farther away and cheaper (Fagnant and Kockelman, 2015). This implies that human drivers can be dropped off right at their final destination without having to search for a spot or having to walk from that spot to their final destination (de Almeida Correia and van Arem, 2016; Fagnant and Kockelman, 2015). Nourinejad and Roorda (2017) model this parking behavior and show that city planners can allocate parking to AVs at locations that are farther away from downtown.

From a technological standpoint, the design and management of car-parks will need to change. Car-parks for regular vehicles are commonly designed according to guidelines published by local governments. The City of Toronto, for example, imposes restrictions on parking space dimensions, orientation of spaces, and the width of the driving aisles (City of Toronto, 2013). Current guidelines are not readily applicable to AVs because they do not consider the possible AV movements within the car-park. Hence, there is a need to initiate a new set of regulations tailored for AV parking.

Although there are currently no guidelines for AV parking design, there are several research streams that exhibit similar properties to the problem at hand. We specify these streams as the following:

- *The stacking problem*: Many storage systems require that items (i.e., any type of merchandise) be stacked on top of each other to increase space efficiency (De Castilho and Daganzo, 1993; Zhang et al., 2002; Jiang and Jin, 2017). In these systems, retrieving a blocked item (i.e., an item that is not on top of the stack) requires some rehandling. Storage operators ideally like to minimize rehandling by optimally stacking items. A similar objective is pursued in AV parking design. We like to fit the AVs in the car-park to minimize the number of relocations whenever retrieving a blocked vehicle.
- *Parking assignment*: Finding a parking space in downtown cores is often time-consuming and difficult. A survey of 20 cities around the world reported that drivers spend 20 minutes on average to find parking (Gallivan, 2011). To address this issue, several studies have focused on modeling search behavior (Boyles et al., 2015; Liu and Geroliminis, 2016; Pel and Chaniotakis, 2017) or developing models that assign parking spaces to vehicles in an optimal fashion (He et al., 2015; Shao et al., 2016; Lei and Ouyang, 2017). Parking assignment also appears in the AV parking design problem. The challenge is to assign the vehicles to the islands of the car-park in a balanced way.
- *Optimal layout design*: Optimal layout design is a problem prevalent in many contexts. The objective is to divide a given geometrical layout into components that serve a purpose. One example is the *Facility Layout Problem* where indivisible departmental components are optimally placed in a given geometrical area to improve production efficiency (Hungerländer and Anjos, 2015; Gonçalves and Resende, 2015). See Anjos and Vieira (2017) for a review of the *Facility Layout Problem*. The AV parking design problem also finds an optimal layout with components that include islands and driving lanes (i.e., gaps) as shown in Fig. 2. By optimally designing these components, we like to improve the efficiency of AV car-parks.

## 3. The model

In this study, we make the following assumptions:

1. *Vehicle size*: All vehicles are assumed to be the same size or at least they are assumed to be smaller than a given vehicle size.
2. *Vehicle control*: We assume that the car-park operator takes control of all AVs that enter the parking facility. The operator can then relocate the vehicles within the facility whenever required. When a vehicle leaves the facility, the control of the AV is directed back to the vehicle owner.
3. *Vehicle arrival*: The vehicles arrive at the car-park following a Poisson distribution with arrival rate $\lambda$ [*veh per h*] and their parking duration (i.e., parking dwell time) follows an exponential distribution with an average parking time of $\mu$ [h]. We also assume that we do not have any information about the actual arrival and departure times of the vehicles.
4. *Exclusivity to autonomous vehicles*: We assume that the car-park is exclusively designed for AVs.

We use the following nomenclature in this study:

**Nomenclature**

*Parameters*

| | |
|---|---|
| $D$ | Design demand [*veh*] |
| $\lambda$ | Arrival rate of vehicles [*veh per h*] |
| $\mu$ | Average parking time [*h*] |
| $L$ | Length of the car-park |
| $W$ | Width of the car-park |
| $l$ | Length of each parking spot |
| $w$ | Width of each parking spot |
| $y$ | Number of vehicle rows in each island |
| $\alpha$ | Width of each driving lane in the inter-island gaps |
| $S$ | Maximum possible number of islands in the car-park |
| $K$ | Maximum number of spots in each stack of the car-park |

*Sets*

| | |
|---|---|
| $I$ | Set of islands |

*Decision variables*

| | |
|---|---|
| $x_i$ | Half the number of columns in island $i$ (non-negative integer variable) |
| $e_i$ | Inter-island gap between island $i-1$ and $i$ allocated to vehicle movements (binary variable) |
| $g_i$ | Inter-island gap between island $i-1$ and $i$ allocated to temporary vehicle storage (non-negative integer variable) |
| $d_i$ | Demand allocated to island $i$ (non-negative variable) |

### 3.1. Geometric properties of the car-park

Consider a rectangular car-park with length $L$ and width $W$. Assume that the car-park has a number of *islands* (See Fig. 2). Each island $i \in I = \{1, 2, \dots, i, \dots, S\}$ has $2x_i$ columns of vehicles: half are parked in the west-bound direction and the other

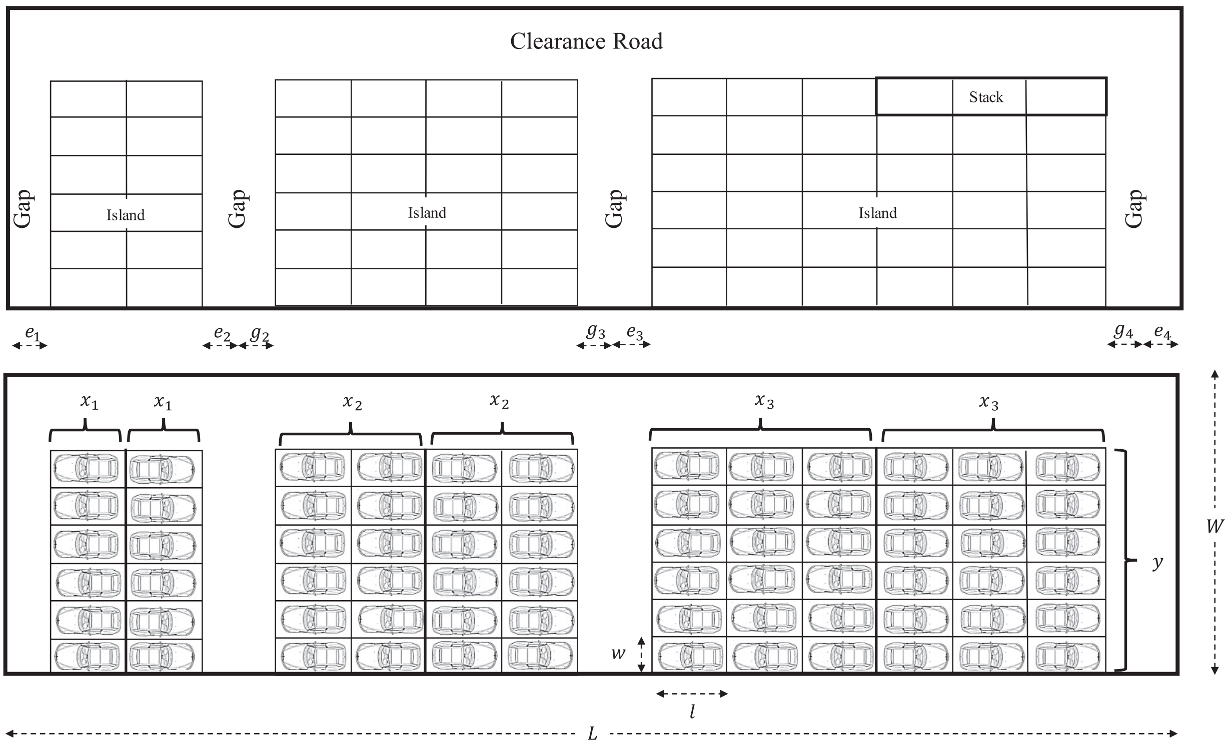


**Fig. 2.** Geometric properties and dimensions of the car-park.

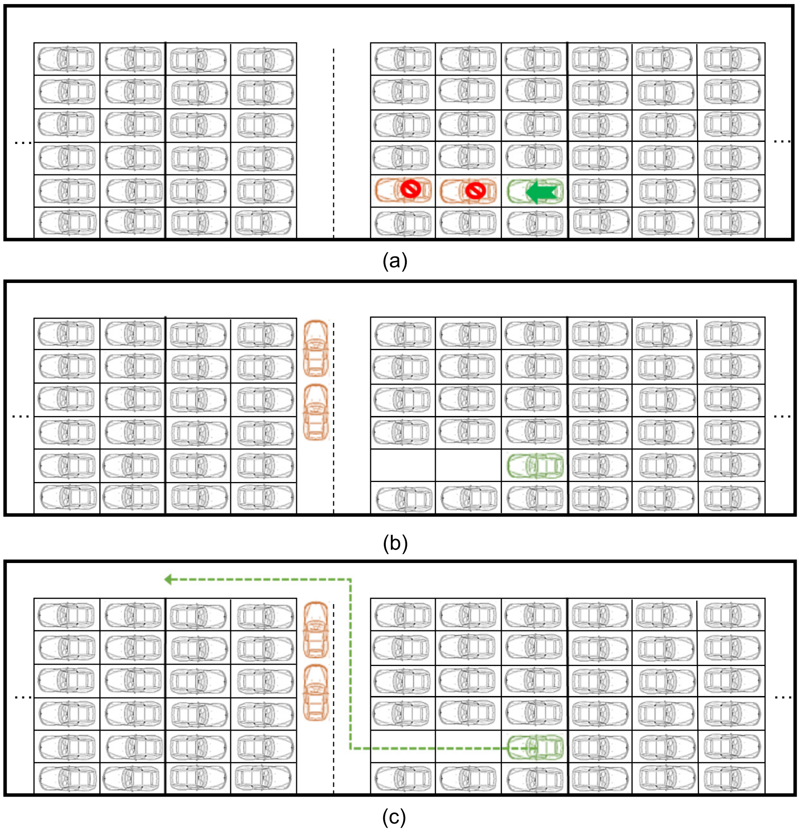


**Fig. 3.** (a) The green vehicle is blocked, (b) the two red vehicles are relocated, (c) the green vehicle exits the car-park. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

half in the east-bound direction. The length of each island is $2lx_i$ where $l$ is the length of each parking spot. The width of each island is $wy$ where $w$ is the width of each parking spot. All islands have $y$ rows of vehicles. Each row of island $i$ has two stacks each containing $x_i$ spots as shown in Fig. 2.

The islands are separated from each other by *gaps* so that the vehicles can maneuver in and out of parking spots. Each gap between island $i-1$ and $i$ has two components: $e_i$ and $g_i$. A minimum gap of $e_i$ is always required between two islands for vehicle mobility and the extra gap $g_i$ is imposed in special cases to store vehicles in the relocation process. The number of gaps is always equal to number of islands plus one. We elaborate the detailed purpose of the gaps in Section 3.2. Finally, there is a *clearance roadway* on the north (or south) of the car-park. This roadway is used for vehicles to enter or leave the car-park. The clearance roadway needs to have direct access to the entrance of the car-park. We assume that $y$ (number of rows) is a given parameter of the model chosen so that the width of clearance roadway $W - yw$ is large enough for vehicle mobility.

### 3.2. Relocation strategy

We implement a relocation strategy that allows any vehicle to leave the car-park at any given point in time so that no vehicle is permanently stranded upon being requested by its owner. Consider the green vehicle in Fig. 3(a) that wants to leave the car-park but is blocked by two red vehicles ahead of it. Our relocation strategy proceeds as follows. We first move the two red vehicles into the inter-island gap to clear space in front of the green vehicle as shown in Fig. 3(b). When cleared, the green vehicle exits the car-park using the clearance roadway as shown in Fig. 3(c).

Recall that the gap between any two islands has two components: $e_i$ and $g_i$. The first gap $e_i$ ensures that there is one lane between islands $i-1$ and $i$ for vehicle mobility. The gap $e_i$ is required whenever island $i-1$ or island $i$ is generated. Formally, $e_i = 1$ if either $x_i \geq 1$ or $x_{i-1} \geq 1$. Otherwise, $e_i = 0$ as stipulated in the following two equations

$$x_{i-1} \leq Me_i \qquad \forall i \in I \cup \{S+1\} \setminus \{1\} \tag{1}$$

$$x_i \leq Me_i \qquad \forall i \in I \tag{2}$$

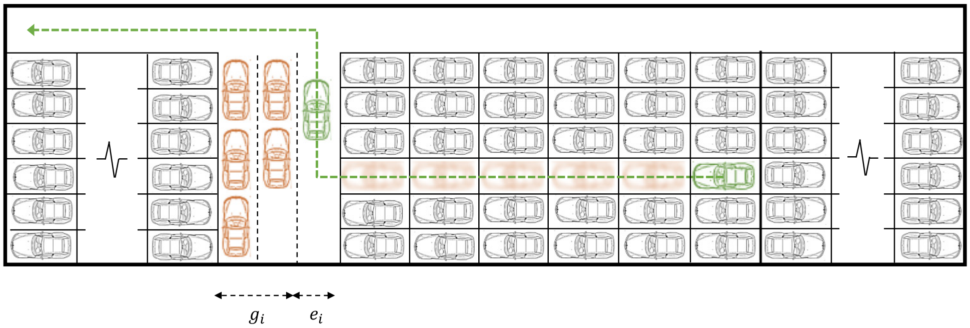


**Fig. 4.** Vehicle relocation with a large inter-island gap. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where $M$ is a sufficiently big number. Note that islands with $x_i = 0$ are virtually not generated. Hence, $S$ (a parameter in the model) is the maximum number of islands that the car-park can hold.

The second gap $g_i$ is the number of lanes needed to temporarily store the blocking (red) vehicles so that any requested (green) vehicle can exit the car-park (see Fig. 3). When the green vehicle leaves the car-park, all the red vehicles return back to their original stack.

The gap $g_i$ between two islands needs to be large enough to store vehicles temporarily. Assume that we want to free the green vehicle in Fig. 4. Here, the gap needs to fit two rows of vehicles. In general form, the gap $g_i$ should be able to hold the maximum number of relocated vehicles which is $x_i - 1$ for island $i$. Hence, we have

$$\frac{ywg_i}{l} \geq x_i - 1 \qquad \forall i \in I \tag{3}$$

where the left-hand-side is the total number of vehicles that can be stored in the gap and the right-hand-side is the maximum number of vehicles that need to be relocated in island $i$.

Given that each gap is shared between two islands, we also have

$$\frac{ywg_i}{l} \geq x_{i-1} - 1 \qquad \forall i \in I \cup \{S+1\} \setminus \{1\}. \tag{4}$$

It is possible that a vehicle cannot immediately park upon its arrival if an ongoing relocation process is not yet completed. Such conflicts can occur when the relocation process is slow. To deal with such situations, the arriving vehicle can temporarily hold back in a designated area until the relocation procedure is completed.

### 3.3. Demand allocation

The car-park is designed to serve a demand of $D$ [*veh*] which is an input of the model. The vehicles enter the car-park following a Poisson distribution with a given arrival rate of $\lambda$ [*veh per h*]. The parking duration follows an exponential distribution with an average parking time of $\mu$ [$h$]. Under steady state conditions, we have $D = \lambda\mu$.

The operator of the car-park decides how to allocate the demand $D$ between the islands to minimize the expected number of relocations. Let $d_i$ (a decision variable) be the demand allocated to island $i$ such that

$$D = \sum_{i=1}^{S} d_i \qquad (\text{demand} - \text{allocation}). \tag{5}$$

Each island's demand $d_i$ must be lower than the supply of parking spots at that island. Recall that island $i$ has $2x_i$ columns and $y$ rows. This leads to a parking supply of $2yx_i$ spots at island $i$ which has to be larger than the demand of vehicles allocated to that island:

$$d_i \leq 2yx_i \qquad \forall i \in I \qquad (\text{supply} - \text{demand}). \tag{6}$$

We assume the demand $d_i$ is randomly assigned between the $2y$ stacks, hence, the demand for each stack is $d_i/(2y)$.

### 3.4. Minimizing expected relocations per vehicle retrieval

We now explain how to find the occupancy probability of each stack using queuing theory. We want to find the probability that a stack with $x_i$ spots is occupied by $v$ vehicles. This queue is described as follows. Assume that each of the $d_i/2y$ vehicles in the stack (of island $i$) are being “served” at the same time. By “serve”, we mean a vehicle reaches its parking duration and needs to leave the car-park (hence, it is served). In total, the number of occupied servers in the queue is equal to the number of vehicles in the stack. The arrival rate of vehicles into island $i$'s stacks is $d_i/2y\mu$ ($\mu$ is the average parking time) and the queuing system remains in steady-state as long as the supply-demand condition in Eq. (6) is justified.

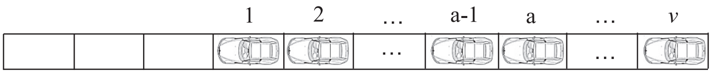


**Fig. 5.** Example of a stack with $v$ vehicles.

The queue is modeled as an $M/M/x_i$ system with a finite system capacity of $x_i$ spots. The state-transition-diagram for this queuing system is presented in Chapter 4.6.4 of Larson and Odoni (1981). This queuing system is technically termed “Erlang’s loss formula”. Accordingly, the probability that each stack of island $i$ holds $v$ vehicles at any given time is $P_{iv}$:

$$P_{iv}(d_i) = \frac{(d_i/2y)^v/v!}{\sum_{t=0}^{x_i}(d_i/2y)^t/t!}. \tag{7}$$

The occupancy probability $P_{iv}$ (also known as the queue length probability) is essential in finding the expected number of relocations in each stack. Let $R_v$ be the number of relocations in island $i$ if there are $v$ vehicles in the stack. Then, the expected number of relocations for taking out any randomly chosen vehicle in the car-park is

$$E[R] = \sum_{i=1}^{S}\sum_{v=0}^{x_i} \frac{d_i}{D}\,\frac{1}{2y}\,P_{iv}(d_i)\,R_v. \tag{8}$$

where the term $d_i/D$ in Eq. (8) is the probability that a random vehicle is chosen in island $i$ and $1/2y$ is the probability that the vehicle is in one of the $2y$ stacks of that island.

To find $R_v$, consider a stack with $v$ vehicles as shown in Fig. 5. The probability that any vehicle is randomly chosen to leave is $1/v$. Assume that vehicle $a$ (where $1 \le a \le v$) in Fig. 5 is randomly chosen to leave the car-park. To discharge vehicle $a$, we first need to relocate $(a-1)$ vehicles that are currently parked ahead of vehicle $a$. These $a-1$ vehicles are moved to the gap. In total, we consider that $a = (a-1)+1$ vehicles need to be moved: $a-1$ vehicle relocations to the gap and the movement of vehicle $a$ itself as well.

Once vehicle $a$ leaves the car-park, the $a-1$ vehicles that are currently in the gap need to retreat back to the stack where they were originally positioned. This leads to an additional $a-1$ vehicle relocations. To summarize, to move vehicle $a$, we need $(a)+(a-1)$ vehicle relocations. Summing this over all $v$ vehicles in the stack and considering that each vehicle has a probability $1/v$ of being chosen to leave, we have

$$R_v = \frac{1}{v}\left[\sum_{a=1}^{v} a + \sum_{a=1}^{v}(a-1)\right] = \frac{1}{v}\left[\frac{v(v+1)}{2} + \frac{(v-1)v}{2}\right] = v \tag{9}$$

Vehicles all have the same probability $1/v$ of leaving because the parking dwell times are memoryless. Hence, the average relocations in a stack with $v$ vehicles is $v$ and the expected number of relocations in the entire car-park (all the islands) is obtained from Eq. (8).

We note that because the demand is stochastic, it is possible that the actual realized demand exceeds the capacity of the car-park which may lead to rejection of some vehicles. This rejection probability can be considered as a penalty in the objective function but it would make the problem intractable. Hence, as our focus is on strategic layout design, we do not consider the rejection probability in the objective function but acknowledge its presence. We discuss the rejection probability in Appendix A and identify the cases where it may be large. We argue that having the rejection probability as an output of the model gives decision-makers an idea of what the ideal demand should be.

As we have used queuing theory to model the car-park, it is important to ensure that the queues are stable and lead to steady-state. To justify the steady-state condition in our M/M/m queues with finite capacity (also known as M/M/m/m), we need to ensure that the demand in each stack $d_i/(2y)$ (of island $i$) is smaller than the number of spots $x_i$ in that island (see (Larson and Odoni, 1981) for the steady-state conditions). This condition is satisfied as long as we impose the supply-demand constraint in Eq. (6) where the equality sign is included for model feasibility. Hence, the queues lead to steady-state.

### *3.5. Mathematical model*

We now present a mixed integer non-linear program to find the optimal layout design of the car-park and the optimal demand allocation between the islands of the car-park.

Before presenting the model, we make two changes in the decision variables to make the problem easier to solve. We replace $x_i$, half the number of columns in island $i$, with $x_{ik}$ which is a binary variable equal to 1 (and otherwise 0) if there are $k$ columns in each stack of island $i$. Similarly, we replace $d_i$ with $d_{ik}$ which is the demand allocated to island $i$ if it holds $k$ parking spots in each stack.

With this change of variable, we have

$$x_i = \sum_{k=0}^{K} k x_{ik} \qquad \forall i \in I$$

where $K$, a parameter of the model, is the maximum number of columns in each stack. To ensure that each stack $i$ takes only one $k$ value, the following constraint is imposed:

$$\sum_{k=0}^{K} x_{ik} = 1 \qquad \forall i \in I.$$

Subsequently, the supply-demand constraint in Eq. (6) changes to

$$d_{ik} \le 2ykx_{ik} \qquad \forall i \in I.$$

The relocation strategy constraints (Eqs. (3) and (4)) become, respectively,

$$\frac{ywg_i}{l} \ge \sum_{k=0}^{K} kx_{ik} - 1 \qquad \forall i \in I$$

and

$$\frac{ywg_i}{l} \ge \sum_{k=0}^{K} kx_{i-1,k} - 1 \qquad \forall i \in I \cup \{S+1\} \setminus \{1\}$$

and the gap constraints (Eqs. (1) and (2)) become, respectively,

$$\sum_{k=0}^{K} kx_{i-1,k} \le Me_i \qquad \forall i \in I \cup \{S+1\} \setminus \{1\}$$

and

$$\sum_{k=0}^{K} kx_{ik} \le Me_i \qquad \forall i \in I.$$

Using the new definition $x_{ik}$, we also change the queue occupancy probabilities $P_{iv}$ to the following. Let $P_{ikv}$ be the probability that a stack of island $i$ with capacity $k$ holds $v$ vehicles. This probability is obtained with minor modification of Eq. (7):

$$P_{ikv}(d_{ik}) = \frac{(d_{ik}/2y)^v/v!}{\sum_{t=0}^{k}(d_{ik}/2y)^t/t!}.$$

The mathematical model, denoted as the *Layout Design* [*LD*] Problem is formally defined as

$$[LD]: \text{ Minimize } \quad E[R] = \sum_{i=1}^{S}\sum_{k=0}^{K}\sum_{v=0}^{k} x_{ik}P_{ikv}(d_{ik})R_v \; d_{ik}/[2yD] \tag{10}$$

subject to

$$D = \sum_{i=1}^{S}\sum_{k=0}^{K} d_{ik} \tag{11}$$

$$d_{ik} \le 2y\ k\ x_{ik} \qquad \forall i \in I, \forall k \tag{12}$$

$$\sum_{k=0}^{K} x_{ik} = 1 \qquad \forall i \in I \tag{13}$$

$$\frac{ywg_i}{l} \ge \sum_{k=0}^{K} kx_{ik} - 1 \qquad \forall i \in I \tag{14}$$

$$\frac{ywg_i}{l} \ge \sum_{k=0}^{K} kx_{i-1,k} - 1 \qquad \forall i \in I \cup \{S+1\}\setminus\{1\} \tag{15}$$

$$\sum_{k=0}^{K} kx_{i-1,k} \le Me_i \qquad \forall i \in I \cup \{S+1\}\setminus\{1\} \tag{16}$$

$$\sum_{k=0}^{K} kx_{ik} \leq Me_i \qquad \forall i \in I \tag{17}$$

$$2l\sum_{i=1}^{S}\sum_{k=0}^{K} kx_{ik} + \alpha\sum_{i=1}^{S+1}(e_i + g_i) \leq L \tag{18}$$

$$e_i \in \{0, 1\} \qquad \forall i \in I \cup \{S+1\} \tag{19}$$

$$g_i, x_{ik} \in \mathbb{Z} \qquad \forall i \in I, \forall k \tag{20}$$

$$x_{ik}, e_i, g_i, d_{ik} \geq 0 \qquad \forall i \in I, \forall k \tag{21}$$

where Eq. (10) minimizes the expected relocations, Eq. (11) is the demand-allocation constraint, Eq. (12) is the supply-demand constraint, Eq. (13) ensures that island $i$'s stacks have $k$ spots, Eqs. (14)–(17) are the inter-island gap constraints, Eq. (18) is the parking dimension constraint, and Eqs. (19)–(21) are the integrality and non-negativity constraints.

The objective function in Eq. (10) quantifies the number of relocations per vehicle retrieval. We can simply convert this function to a rate of relocations (i.e., relocations per h), if we multiply the objective function (10) by the arrival rate of vehicles $\lambda$ [*vehicles per h*] (which is equal to the departure rate of vehicles). Given that $\lambda$ is a parameter in the model, it can be factored out from the objective function. Hence, changing the unit of the objective function from [*relocations per vehicle*] to [*relocations per h*] does not change the outputs of the model.

Constraint (18) is the parking dimension constraint which indicates that the width of all islands (first term on left-side) and the width of all gaps (second term on left-side) must not exceed the length of the car-park $L$. In this constraint, the width of gap $i$ is $\alpha.(e_i + g_i)$ where $\alpha$ is the required width of each travel-line in the gap. $\alpha$ is a function of the turning radius of the vehicles and the preferred safety margin in the car-park to avoid incidents while moving the AVs. Technically, $\alpha$ must satisfy $\alpha > w$ because the width of the gap needs to be larger than the width of the AVs so that the AVs do not crash when they drive in the gap lanes.

The presented mathematical model is difficult to solve because it is a mixed integer program with a non-linear objective function. Because of the structure of the model, existing commercial optimization software cannot be used especially if an exact solution is of interest. Hence, to solve [*LD*], we present a custom-made exact algorithm that is based on generating Benders decomposition cuts. Given that the exact algorithm has a high computation time in several instances, we also present a heuristic algorithm that is much faster.

## 4. Solution methodology

### *4.1. An exact decomposition algorithm*

Using Benders decomposition, we divide the problem into a *master-problem* [*MP*] and a *sub-problem* [*SP*] (Geoffrion, 1972). In the sub-problem, it is assumed that the layout of the car-park is fixed and equal to $\bar{x}_{ik}, \forall i \in I$. With the layout $\bar{x}_{ik}$ fixed, the sub-problem (which is a function of $\bar{x}_{ik}$) finds the optimal allocation of the demand $D$ between the islands. The sub-problem is defined as

$$[SP(\bar{x}_{ik})] : \text{Minimize} \quad E[R] = \sum_{i=1}^{S}\sum_{k=0}^{K}\sum_{v=0}^{k} \bar{x}_{ik}P_{ikv}(d_{ik})R_v \ d_{ik}/[2yD] \tag{22}$$

subject to

$$d_{ik} \leq 2yk\ \bar{x}_{ik} \qquad \forall i \in I, \forall k \qquad (\text{supply} - \text{demand}) \tag{23}$$

$$D = \sum_{i=1}^{S}\sum_{k=0}^{K} d_{ik} \qquad (\text{demand} - \text{allocation}) \tag{24}$$

$$d_{ik} \geq 0 \qquad \forall i \in I, \forall k \tag{25}$$

which is a non-linear mathematical model with linear constraints. The objective function in Eq. (22) minimizes expected relocations for a given set of $\bar{x}_{ik}$, Eq. (23) is the supply-demand constraint, Eq. (24) is the demand-allocation constraint, and Eq. (25) ensures non-negativity of the decision variables. The uniqueness of the sub-problem's solution depends on the convexity of the objective function in Eq. (22) which is discussed in the next subsection. Whenever we solve the sub-problem, we obtain an upper-bound on the optimal solution of the original problem [*LD*]. This upper-bound decreases and gets closer to the optimal solution (of [*LD*]) as we try out better (more efficient) feasible layouts $\bar{x}_{ik}, \forall i \in I$.

The purpose of the master-problem is to iteratively generate different layouts (i.e., $x_{ik}$, $g_i$, $e_i$) until the best layout is found. Every time a new layout is obtained in the master-problem, the sub-problem is solved to allocate the demand $D$ to the islands of the newly generated layout. In the master-problem, we implicitly consider the supply-demand constraints (Eq. (23)) in the form of Benders cuts. The supply-demand constraint is the only constraint that links the layout design variables $x_{ik}$, $e_i$, $g_i$ to demand allocation variables $d_{ik}$.

The master-problem gives a lower-bound on the optimal objective function of the original problem [*LD*] because the supply-demand constraints (Eq. (23)) are relaxed and presented as Benders cuts in the master-problem. Absence of these constraints indicates that the objective of the master-problem is always better (lower) than the true objective of [*LD*]. Hence, the master-problem provides a lower-bound for the original problem. The sub-problem's solution is accounted in the master-problem using Benders cuts constraints.

The master-problem is presented as follows

$$[MP] : \text{Minimize } Z \tag{26}$$

subject to

$$Z \geq \sum_{i=1}^{S} \sum_{k=0}^{K} \sum_{v=0}^{k} x_{ik} P_{ikv}(\bar{d}_{ikt}) R_v \ \bar{d}_{ikt}/[2Dy] - u_{ikt} \ 2yk \ x_{ik} \qquad \forall i \in I, \forall t \in T, \forall k \tag{27}$$

$$\text{Constraints } (13) - (20)$$

$$x_{ik}, e_i, g_i \geq 0, \qquad \forall i \in I, \forall k \tag{28}$$

where $Z$ is the lower-bound currently obtained, $\bar{d}_{ikt}$ is the solution of the sub-problem at iteration $t$ (i.e., cut $t$), $T$ is the set of Benders cuts, and $u_{ikt}$ is the Lagrange multiplier of Eq. (23) when the sub-problem was solved in the $t$th iteration. The presented master-problem is a mixed integer-linear program which can be solved using available commercial software.

We summarize the steps of the exact algorithm as the following:

1. *Initialize*: Find a feasible layout using an integer program with any objective function subject to Constraints (13) to (20) and $x_{ik}$, $e_i$, $g_i \geq 0$, $\forall i \in I$, $\forall k$. Solve the sub-problem and generate Benders cuts $T$. Let $UB$ be the currently found upper-bound which is the objective function of the sub-problem.
2. *The master-problem*: Solve the master-problem using the Set $T$ of all generated Benders cuts so far. Let $LB$ be the currently found lower-bound which is the objective function $Z$ of the master-problem.
3. *The sub-problem*: Solve the sub-problem using the most recent layout obtained from the master-problem in Step 2. Update $UB$ if the sub-problem has found a smaller upper-bound than the one currently available. Add the new Benders cuts to Set $T$.
4. *Terminate*: Terminate the algorithm if the upper-bound and the lower-bound are close enough to each other such that $UB - LB < \epsilon$, where $\epsilon$ is a predefined threshold value. Go to Step 2 if the termination condition is not satisfied.

### *4.2. Solving the sub-problem*

We now analyze the solution of the sub-problem. The sub-problem is a minimization problem subject to a convex and compact feasible solution space because the solution space is a set of linear inequalities. By proving that the objective function of the sub-problem is convex, we conclude that the sub-problem has a unique solution (Bazaraa et al., 2013). The convexity of the objective function also indicates that we can use any standard convex minimization method to solve the sub-problem. We now prove that the sub-problem is convex in the following Lemma:

**Lemma 1.** *The objective function of the sub-problem is convex.*

**Proof.** Let us first rewrite the objective function of the sub-problem by moving some terms around

$$E[R] = \sum_{i=1}^{S} \sum_{k=0}^{K} \bar{x}_{ik}/D \sum_{v=0}^{k} P_{ikv}(d_{ik}) R_v \ d_{ik}/[2y]$$

where $\bar{x}_{ik}/(2yD)$ are fixed parameters. Notice that $E[R]$ is now a sum of functions

$$Q_{ik}(d_{ik}) = \sum_{v=0}^{k} P_{ikv}(d_{ik}) R_v \ d_{ik}/[2y]$$

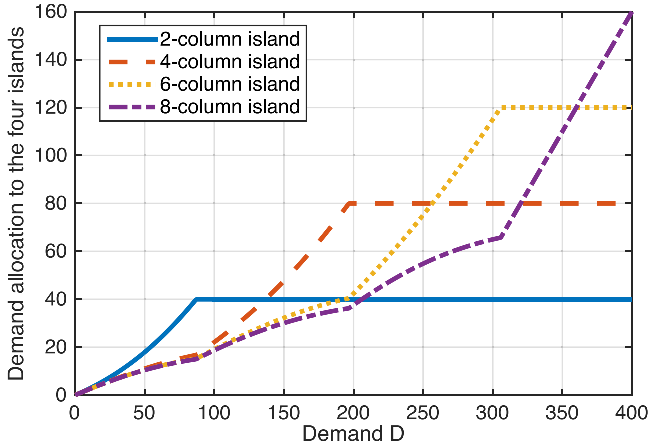


**Fig. 6.** Solution of the sub-problem for demand $D = 0$ to $D = 400$.

multiplied by fixed parameters. To show that $E[R]$ is convex, we need to show that each function $Q_{ik}(d_{ik})$ is convex. This holds true because sum of a set of convex functions is convex as well. Let us first simplify $Q_{ik}(d_{ik})$ to

$$\begin{aligned} Q_{ik}(d_{ik}) &= \sum_{v=0}^{k} P_{ikv}(d_{ik}) R_v \ d_{ik}/[2y] \\ &= \sum_{v=0}^{k} \frac{(d_{ik}/2y)^v/v!}{\sum_{t=0}^{k}(d_{ik}/2y)^t/t!} \ v \ d_{ik}/[2y] \\ &= \sum_{v=0}^{k} \frac{(d_{ik}/2y)^{v+1}/(v-1)!}{\sum_{t=0}^{k}(d_{ik}/2y)^t/t!}. \end{aligned}$$

We now show that $Q_{ik}(d_{ik})$ is convex for any given $i$ and $k$. We do this by exhaustively enumerating $Q_{ik}(d_{ik})$ for all $k$. Assuming $k = 1$ leads to

$$Q_{ik}(d_{ik}) = \frac{d_{ik}^2}{2y(d_{ik} + 2y)} \qquad k = 1.$$

Taking the second derivative of the above, we have

$$\frac{\partial^2 Q_{ik}(d_{ik})}{\partial d_{ik}^2} = \frac{4y}{(d_{ik} + 2y)^3} > 0 \qquad k = 1$$

which is strictly positive indicating that $Q_{ik}(d_{ik})$ is convex for $k = 1$.

Using the same procedure for $k = 2$ and $k = 3$, the second derivatives of $Q_{ik}(d_{ik})$ are, respectively,

$$\frac{\partial^2 Q_{ik}(d_{ik})}{\partial d_{ik}^2} = \frac{32y^2(3d_{ik}^2 + 12d_{ik}y + 8y^2)}{(d_{ik}^2 + 4d_{ik}y + 8y^2)^3} > 0 \qquad k = 2$$

and

$$\frac{\partial^2 Q_{ik}(d_{ik})}{\partial d_{ik}^2} = \frac{12y\left(-d_{ik}^6 + 216d_{ik}^4y^2 + 1632d_{ik}^3y^3 + 5184d_{ik}^2y^4 + 6912d_{ik}y^5 + 4608y^6\right)}{\left(d_{ik}^3 + 6d_{ik}^2y + 24d_{ik}y^2 + 48y^3\right)^3} > 0 \qquad k = 3. \tag{29}$$

Eq. (29) is positive for $d_{ik} > 0$ and $y > 0$ such that the supply-demand constraint $d_{ik}/2y \leq k$ ($k = 3$) is justified. Continuing the same process for other $k$, we observe $\partial^2 Q_{ik}(d_{ik})/\partial d_{ik}^2 > 0$. We conclude that the objective function of the sub-problem is convex. □

Let us now illustrate an example to better elaborate the sub-problem. Consider four islands that have 2, 4, 6, and 8 columns. All islands have 20 rows, i.e., $y = 20$. We increase the total demand $D$ and illustrate how it is allocated to the islands. As shown in Fig. 6, all islands are initially filled up but at different rates. The smaller islands are filled first and larger islands are filled last according to our demand allocation strategy.

### 4.3. A heuristic model

Solving the exact algorithm can be time consuming in cases where there are abundant geometrical combinations that could fit the car-park. To circumvent this high computation time, a heuristic algorithm is presented that provides an upper-bound solution in a much faster computation time. The idea of the heuristic algorithm is as follows. To minimize relocations, we start off by fitting the demand $D$ into two-column islands (which is similar to currently available parking facilities for regular non-automated vehicles). If the demand is too large to fit into two-column islands, we test out three-column islands and we continue the process until we find a layout that is able to serve the demand. As we increase the number of columns in the islands (i.e., as the islands become bigger), the cost of relocation increases. Hence, we stop at the smallest number of columns that provides sufficient space for all the vehicles.

We summarize the steps of the heuristic algorithm in the following:

1. *Form two-column islands*: Set $x_i = 1$ so that each island has two columns.
2. *Find the island composition*: Set $s = \lceil \frac{D}{2x_i} \rceil$. If $D$ is divisible by $2x_i$, then each island holds $2x_i$ columns. Otherwise, there are $s-1$ islands with $2x_i$ columns and one island with $2\lceil \frac{D-2(s-1)x_i}{2} \rceil$ columns.
3. *Find the inter-island gaps*: Calculate $g_i$ and $e_i$ by solving an integer program with any objective function subject to Constraints (14)–(18) with fixed $x_i$ from Step 2.
4. *Feasibility check*: It is possible that the current island size $x_i$ cannot fit the car-park dimensions when we include the gaps. This leads to infeasibility of the integer program from Step 2. If a feasible layout is not obtained, set $x_i = x_i + 1$ and go back to Step 2. Otherwise, proceed to Step 5.
5. *Outputs*: Calculate the objective function $E[R]$ based on the currently obtained solution.

When the demand is not divisible by the number of columns, Step 2 of the algorithm finds a composition consisting of $s$ islands with the same number of columns and one last smaller island. The feasibility of the compositions is checked in Steps 3 and 4. The maximum number of iterations in the heuristic is $\lceil D/2y \rceil$ which happens when the algorithm goes through all the different layout compositions until it checks the largest possible island. In some cases, it is possible that no feasible layout is obtained in the heuristic algorithm because the heuristic only checks a subset of the entire solution space. In such cases, the exact algorithm provides the only solution to the problem.

### 4.4. Measures of effectiveness

We present the following measures to assess and compare the results of the model. The measures of effectiveness are the following:

- *Expected relocations*: This is the primary measure indicating the number of relocations per vehicle retrieval, i.e., the objective function of [*LD*].
- *Parking supply*: Parking supply is the number of spots in all islands to serve a given demand $D$.
- *Utilization*: Utilization is the ratio of the total area allocated to the islands over the area of the entire car-park. It is clear that utilization is never equal to 1 because some of the land is allocated to the gaps and the clearance roadway. It is also clear that utilization increases with demand because the operator has to turn more of the car-park into islands.
- *Maximum demand*: Maximum demand $D_{\max}$ is the largest demand that can be served by a given car-park. To find $D_{\max}$, we increase the demand $D$ incrementally to the point where the exact algorithm can no longer find a feasible solution to serve the demand.
- *Spatial efficiency ratio*: This measure is the ratio of $D_{\max}$ over the maximum possible demand if we only have two-column islands. Informally, this ratio indicates the economic benefit of turning existing conventional car-parks for regular vehicles into fully-automated AV parking facilities. As an example, a spatial efficiency ratio of 2 indicates that we can fit twice as many AVs than regular vehicles in the same car-park. For the two-column islands we use spot dimensions $l = 5$ and $w = 2.8$ which are the parking dimensions of a regular vehicle. For AV parking spots, we use $l = 5$ and $w = 2$.

## 5. Numerical experiments

We perform the following numerical experiments to gain managerial insights on AV parking operations and to assess the computational efficiency and accuracy of the two solution algorithms. We set the termination threshold to $\epsilon = 0.05$ in the exact algorithm. To solve the master-problem integer program, we use the ILOG CPLEX package. For the sub-problem, we use a *Sequential Quadratic Programming* algorithm to find the optimal solution.

### 5.1. Parking demand

Consider a car-park with dimensions $L = 150$ [$m$], $W = 65$ [$m$], $l = 5$ [$m$], and $w = 2$ [$m$]. There are $y = 30$ rows in each island and the width of the *clearance roadway* is 5 [$m$]. We increase the demand from $D = 600$ [*veh*] to $D = 780$ [*veh*] and depict the geometrical shape of the optimal car-park in Fig. 7. When we establish a low demand, the islands all have only two columns which is very similar to existing parking facilities for regular vehicles. The two-column design has the lowest relocation cost and is desirable when demand is low to medium. As we impose a higher demand, the islands become bigger with more columns. The two-column islands are eliminated at the largest demands because these islands require gaps that take up valuable land that could otherwise be used as island space. The maximum demand $D_{\max}$ that the car-park can serve is 780 [*veh*]. The parking supply in the six instances is 660, 660, 720, 720, 780, and 780, respectively.

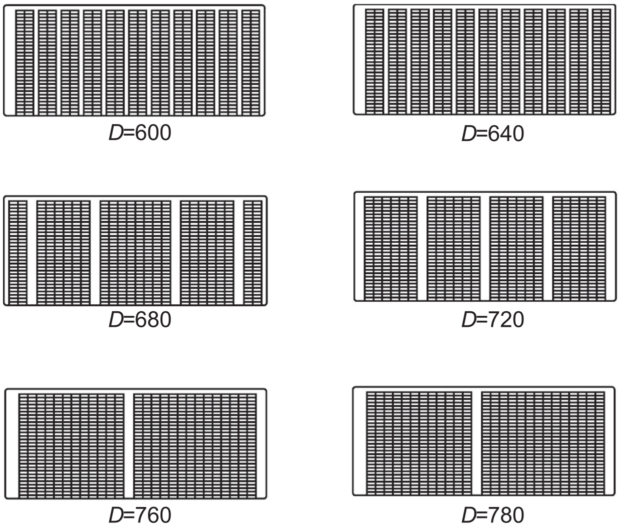


**Fig. 7.** Optimal layout at $D = 600, 640, 680, 720, 760, 780$.

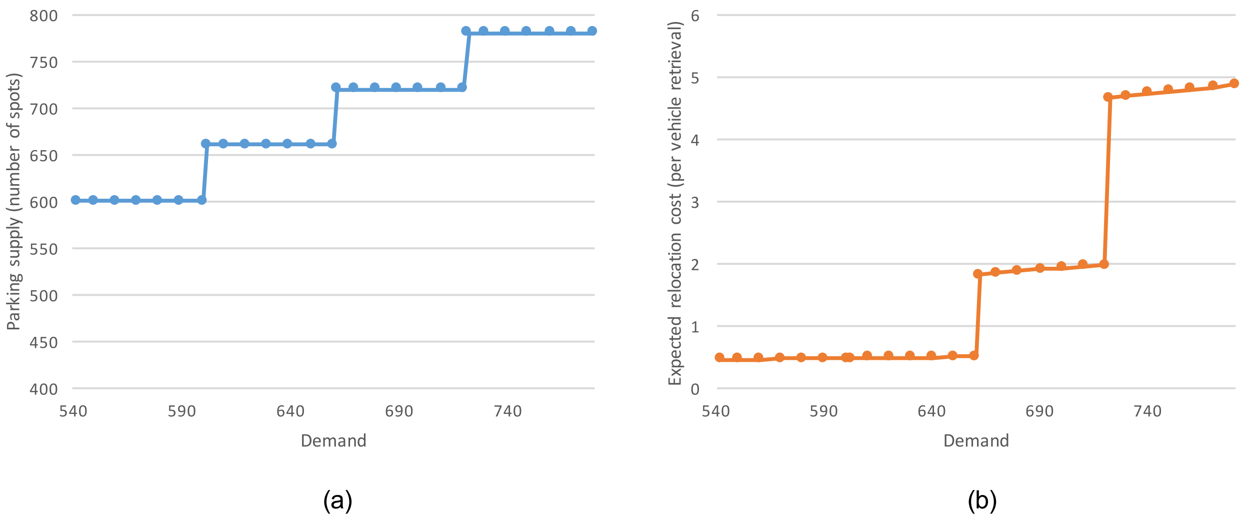


**Fig. 8.** (a) Parking supply and (b) expected relocation cost at different demand $D$.

We present the parking supply (i.e., number of spots) and relocation cost (i.e. $E[R]$) with respect to demand in Fig. 8. It is shown that parking supply increases in a step-wise fashion. The points where the steps occur are the points where the layout changes radically. For instance, from $D = 0$ to $D = 660$, the optimal layout is 11 two-column islands as shown in Fig. 7 ($D = 600$) where the demand is evenly distributed between the islands. At $D = 661$, however, the car-park layout changes radically which leads to a jump in parking supply. Fig. 8 illustrates that for the highest demand $D = 780$ [$veh$], we need to approximately relocate 5 vehicles at any random retrieval.

For expected relocations $E[R]$, we see the same step-wise jumps where the radical layout changes occur. However, there is also a gradual increase in every step which is intuitive as a higher demand requires more relocations. The insight here is that operators need to choose their design demand $D$ while considering the jumps in the relocation cost since a marginal decrease in $D$ can substantially reduce the relocation cost.

### 5.2. Maximum demand

The maximum demand $D_{\max}$ depends on the area and dimensions of the car-park. We present 13 car-park dimensions in Table 1. The instances have different length $L$ and width $W$ but their area is fixed, i.e., $LW = 6890[m^2]$ is constant. The highest maximum demand is $D_{\max} = 560$ which occurs in the square orientation with $L \approx W$ (Instance 7). This square orientation also has the lowest relocation cost among the other instances which makes it a desirable orientation. We depict the layout

**Table 1**
Layout of 13 instances. *Key*: (2,10×5) means there are 5 islands with 10 columns and one island with 2 columns.

| Instance | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| y | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 |
| W | 23 | 33 | 43 | 53 | 63 | 73 | 83 | 93 | 103 | 113 | 123 | 133 | 143 |
| L | 299.5 | 208.8 | 160.2 | 130.0 | 109.3 | 94.4 | 83.0 | 74.1 | 66.9 | 61.0 | 56.0 | 51.8 | 48.2 |
| A | 6890 | 6890 | 6890 | 6890 | 6890 | 6890 | 6890 | 6890 | 6890 | 6890 | 6890 | 6890 | 6890 |
| Maximum demand $D_{\max}$ | 520 | 540 | 560 | 550 | 540 | 560 | 560 | 540 | 500 | 440 | 480 | 390 | 420 |
| Objective function | 3.46 | 3.96 | 5.26 | 4.03 | 3.21 | 6.12 | 5.26 | 4.41 | 2.31 | 0.50 | 0.50 | 0.50 | 0.50 |
| Utilization (%) | 75.48 | 78.39 | 81.29 | 79.84 | 78.39 | 81.29 | 81.29 | 78.39 | 72.58 | 63.87 | 69.68 | 56.61 | 60.97 |
| Spatial efficiency ratio | 1.69 | 1.80 | 1.67 | 1.80 | 1.61 | 1.60 | 1.67 | 1.69 | 1.79 | 1.41 | 1.43 | 1.41 | 1.40 |
| Layout | 2, 10×5 | 2, 10, 12×2 | 14×2 | 10, 12 | 8, 10 | 16 | 14 | 12 | 2, 8 | 2×4 | 2×4 | 2×3 | 2×3 |

**Table 2**
Impact of gap width $\alpha$ on the layout.

| Instance | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Gap width $\alpha$ | 2 | 2.2 | 2.4 | 2.6 | 2.8 | 3.0 | 3.2 | 3.4 | 3.6 | 3.8 | 4.0 |
| Maximum demand $D_{\max}$ | 840 | 780 | 780 | 780 | 780 | 780 | 780 | 780 | 780 | 780 | 780 |
| Objective function | 9.79 | 2.58 | 3.07 | 4.10 | 4.10 | 4.86 | 4.86 | 8.91 | 8.91 | 8.91 | 8.91 |
| Utilization (%) | 0.86 | 0.80 | 0.80 | 0.80 | 0.80 | 0.80 | 0.80 | 0.80 | 0.80 | 0.80 | 0.80 |
| Spatial efficiency ratio | 1.67 | 1.55 | 1.69 | 1.69 | 1.69 | 1.69 | 1.69 | 1.86 | 1.86 | 1.86 | 1.86 |
| Layout | 2, 26 | 2, 8×3 | 8×2, 10 | 2, 12×2 | 2, 12×2 | 12, 14 | 12, 14 | 2, 24 | 2, 24 | 2, 24 | 2, 24 |

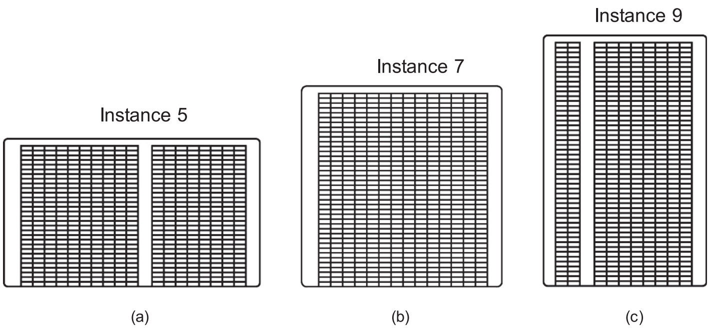


**Fig. 9.** Optimal layout of Instance 5, 7, and 9 in Table 1.

of three instances in Fig. 9 where Instance 5 is elongated horizontally, Instance 7 is a square, and Instance 9 is elongated vertically.

We now fix the demand at $D = D_{\max}$ for each instance and obtain the other measures of effectiveness which are presented in Fig. 10. We perform this for three separate rectangle areas, i.e., $LW$ is fixed in each car-park. Fig. 10a and 10b show that $D_{\max}$ and the utilization ratio are both generally higher in the square orientation where $W = L$. The highest utilization is between 75% and 80% in the three cases indicating that 80% of the land is allocated to the parking spots and the remaining 20% is allocated to the clearance roadway and the inter-island gaps.

Fig. 10c shows that the spatial efficiency is on average 1.62 but can be as high as 1.87 indicating that AV parking facilities can store 87% more vehicles than conventional parking facilities. When the car-park width $W$ is large, the spatial efficiency occurs only because each AV parking spot takes smaller space than a regular parking spot (with dimensions $w = 2.8$ and $l = 5$). For a smaller car-park width, however, the spatial efficiency happens due to spot dimensions and vehicle relocation.

Finally, for each of the instances, Fig. 10d shows the relocation cost versus $D_{\max}$. This highlights the pareto efficiency in $D_{\max}$ and $E[R]$ for the square layouts.

### 5.3. Impact of gap width on layout

We analyze the impact of the gap width $\alpha$ on the optimal layout of the car-park. The measures of effectiveness are presented for 11 instances with different $\alpha$ in Table 2. The utilization ratio decreases with $\alpha$ because the gaps take up a longer width and less space is allocated to the islands. For the same reason, the maximum demand $D_{\max}$ also decreases with $\alpha$ as fewer vehicles can be fit in the car-park. The objective function $E[R]$ first experiences a plunge from Instance 1 to Instance 2 because Instance 2 fits a smaller number of vehicles. From Instance 2 onward, however, the objective function $E[R]$ strictly increases because although we are still serving the same demand $D_{\max} = 780$ [$veh$], we have to fit them in less efficient layouts.

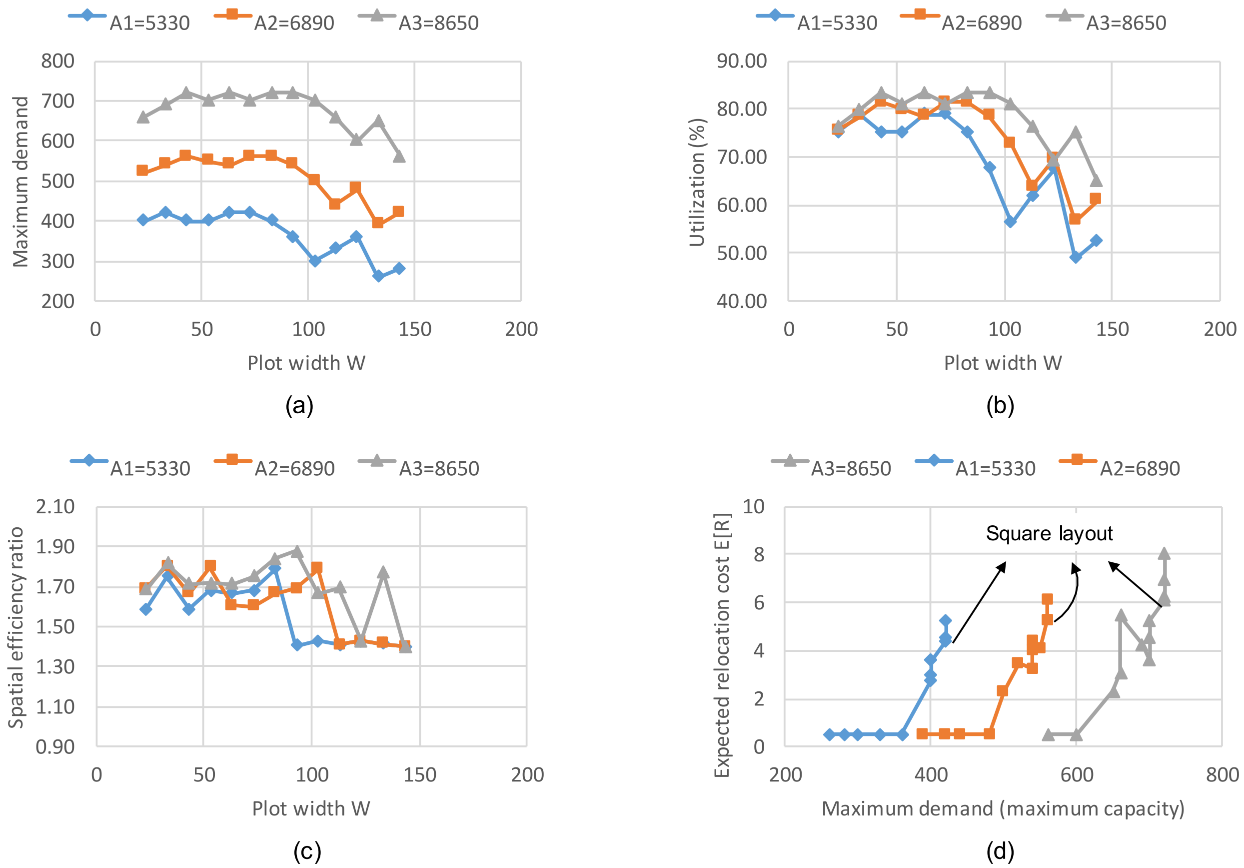


**Fig. 10.** (a) Maximum demand, (b) utilization, (c) spatial efficiency ratio, and (d) expected relocation cost.

Finally, recall that the spatial efficiency ratio is the ratio of $D_{\max}$ to maximum demand that fits in a two-column design. As we increase $\alpha$, fewer vehicles fit in the two-column design but $D_{\max} = 780$ remains constant from Instance 2 to 11. Hence, the spatial efficiency ratio increases from 1.55 (Instance 2) to 1.86 (Instance 11).

### 5.4. Heuristic algorithm

The heuristic algorithm is compared to the exact algorithm for 18 different test instances as shown in Table 3. We let the exact algorithm continue running until we see convergence between the upper and lower bounds with the termination threshold of $\epsilon = 0.05$. The longest computation time occurs in Instance 15 which takes 5.3 [h] to run in the exact algorithm but 0.4 [s] in the heuristic. Clearly, the objective function of the exact algorithm is always lower or at least equal to the heuristic. In Instance 1, we observe that the exact algorithm's objective function is 18% lower than the heuristic. This happens as the heuristic fully fills up 13 two-column islands (until the entire demand allocated) whereas the exact algorithm partially fills up 15 two-column islands. Hence, the heuristic is not strong in exploring cases of partial filling. Since the heuristic algorithm does not consider the objective function and it is terminated whenever it finds a feasible layout, it does not consider the possibility of making the gaps smaller to add other islands and decrease the objective function. Therefore, it creates larger gaps in instances 1, 4, 8, and 13 in comparison to the exact solutions. In Instance 17, we also observe a 15% difference in objective functions because the exact algorithm is able to explore a wider range of possible layout compositions compared to the heuristic. The same condition applies in Instance 15 for which we illustrate the exact and the heuristic layouts in Fig. 11. Finally, in Instance 9, we observe that the two algorithms provide the same objective but the exact algorithm takes 229.62 [*s*] to solve which is significantly longer than the computation time of the heuristic which is only 0.42 [*s*]. This indicates that the heuristic is able to find the optimal answer in a shorter time in some instances.

## 6. Conclusions

AVs could have a great impact on the design of car-parks in the future. While existing parking facilities have islands with only two rows of vehicles, future designs tailored for AVs can have multiple rows of vehicles stacked behind each other. The multi-row design can lead to blockage of some vehicles which can be handled if the operator moves the vehicles in

**Table 3**
Comparison between exact and heuristic methods. Key: A layout (2 × 3,8) indicates that there are 3 islands with 2 columns and one islands with 8 columns.

| Instance | y | W | L | A | D | Exact | | | Heuristic | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Objective function | CPU time [$s$] | Layout | Objective function | CPU time [$s$] | Layout |
| 1 | 20 | 43 | 100 | 4300 | 200 | 0.4167 | 4.96 | 2 × 7 | 0.5 | 0.38 | 2 × 5 |
| 2 | 20 | 43 | 100 | 4300 | 280 | 0.5 | 27.85 | 2 × 7 | 0.5 | 0.40 | 2 × 7 |
| 3 | 20 | 43 | 100 | 4300 | 320 | 2.7573 | 13.44 | 8 × 2 | 2.7573 | 0.41 | 8 × 2 |
| 4 | 30 | 63 | 130 | 8190 | 480 | 0.4706 | 71.40 | 2 × 9 | 0.5 | 0.40 | 2 × 8 |
| 5 | 30 | 63 | 130 | 8190 | 540 | 0.5 | 159.05 | 2 × 9 | 0.5 | 0.41 | 2 × 9 |
| 6 | 30 | 63 | 130 | 8190 | 600 | 1.3769 | 1211.08 | 2 × 4, 6 × 2 | 1.8154 | 0.42 | 2, 6 × 3 |
| 7 | 30 | 63 | 130 | 8190 | 660 | 4.031 | 118.85 | 10, 12 | 4.031 | 0.43 | 10, 12 |
| 8 | 30 | 63 | 150 | 9450 | 500 | 0.431 | 303.31 | 2 × 11 | 0.4808 | 0.40 | 2 × 9 |
| 9 | 30 | 63 | 150 | 9450 | 630 | 0.4884 | 229.62 | 2 × 11 | 0.4884 | 0.42 | 2 × 11 |
| 10 | 30 | 63 | 150 | 9450 | 690 | 1.8983 | 4523.56 | 2 × 2, 6 × 2, 8 | 1.9236 | 0.43 | 6 × 4 |
| 11 | 30 | 63 | 150 | 9450 | 720 | 1.9615 | 2065.08 | 6 × 4 | 1.9615 | 0.43 | 6 × 4 |
| 12 | 30 | 63 | 150 | 9450 | 750 | 4.7623 | 472.06 | 12,14 | 4.7623 | 0.53 | 12,14 |
| 13 | 40 | 83 | 200 | 16,600 | 1000 | 0.4545 | 13082.17 | 2 × 15 | 0.4902 | 0.43 | 2 × 13 |
| 14 | 40 | 83 | 200 | 16,600 | 1200 | 0.5 | 11254.24 | 2 × 15 | 0.5 | 0.45 | 2 × 15 |
| 15 | 40 | 83 | 200 | 16,600 | 1240 | 1.7804 | 18956.23 | 2 × 3, 6 × 3, 8 | 1.839 | 0.54 | 2, 6 × 5 |
| 16 | 40 | 83 | 200 | 16,600 | 1280 | 1.8702 | 12426.87 | 2, 6 × 5 | 1.8702 | 0.47 | 2, 6 × 5 |
| 17 | 40 | 83 | 200 | 16,600 | 1360 | 2.998 | 10951.65 | 8 × 3, 10 | 3.2962 | 0.49 | 4, 10 × 3 |
| 18 | 40 | 83 | 200 | 16,600 | 1440 | 6.9813 | 6740.86 | 18 × 2 | 6.9813 | 0.51 | 18 × 2 |

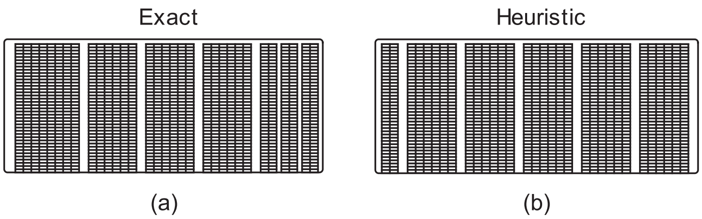


**Fig. 11.** (a) Exact and (b) heuristic layout for Instance 15 in Table 3.

the facility in an optimal way. Naturally, the number and pattern of relocation changes with the layout of the car-park. This paper investigates the problem of finding the optimal car-park layout design that minimizes relocations while fitting a given number of vehicles in the car-park.

Finding the optimal layout can have many social benefits. For one, we find that AV car-parks can decrease the need for parking space by an average of 62% and a maximum of 87%. This revitalization of space that was previously used for parking can lead to social benefits when car-parks are converted into commercial and residential land-uses. Second, our study captures the influential factors that impact the optimal layout of car-parks for AVs. For instance, we show that square-shaped car-parks can enhance several measures of effectiveness. Finally, we capture the impact of the demand (for parking) on the optimal car-park layout. We show that when demand is low, the car-park has only two-column islands and when demand is high, the optimal layout becomes more complex.

This study focuses on the high-level strategic design of car-parks by finding the optimal layout under several assumptions that are valid for higher level planning. For finer level operational planning, however, we need to make adjustments to the model. We name some of these modifications that require further research.

For finer level operational planning, a new model is required that considers individual characteristics of each vehicle including arrival time, planned departure time, and vehicle size. Knowledge of departure times can significantly influence how the vehicles are arranged in the car-park. Ideally, the vehicles with earlier departure times should not be buried deep in the islands. They should instead be on top of the islands for a fast retrieval when they are summoned. Users can provide their departure times to the car-park operator using any smart platforms such as a mobile app.

In the model, we assume that parking demand is constant and fixed throughout the planning period which leads to one optimal layout for the car-park. In practice, however, this optimal layout can change within the day according to dynamic parking demand. For example, the facility can have one layout in the morning and another layout in the afternoon. In future research, it would be valuable to derive the optimal dynamic layout of the car-park according to changes in demand.

While we have proposed one of the first layouts for AV car-parks, other potential layouts could be tested as well to assess whether the parking footprint and the number of relocations can be further reduced. We present a model for cases where the car-park is a rectangle with given dimensions. A clear-cut rectangular car-park may not always exist in reality. Hence, it is important to extend the model to solve other irregular car-park profiles as well. It is also important to consider the optimal layout in multi-storey buildings where the floors are connected either through a elevators (for vehicles) or ramps. Lastly, fork-lift systems can be used as a means of relocating regular vehicles in a car-park. This allows one to manage a

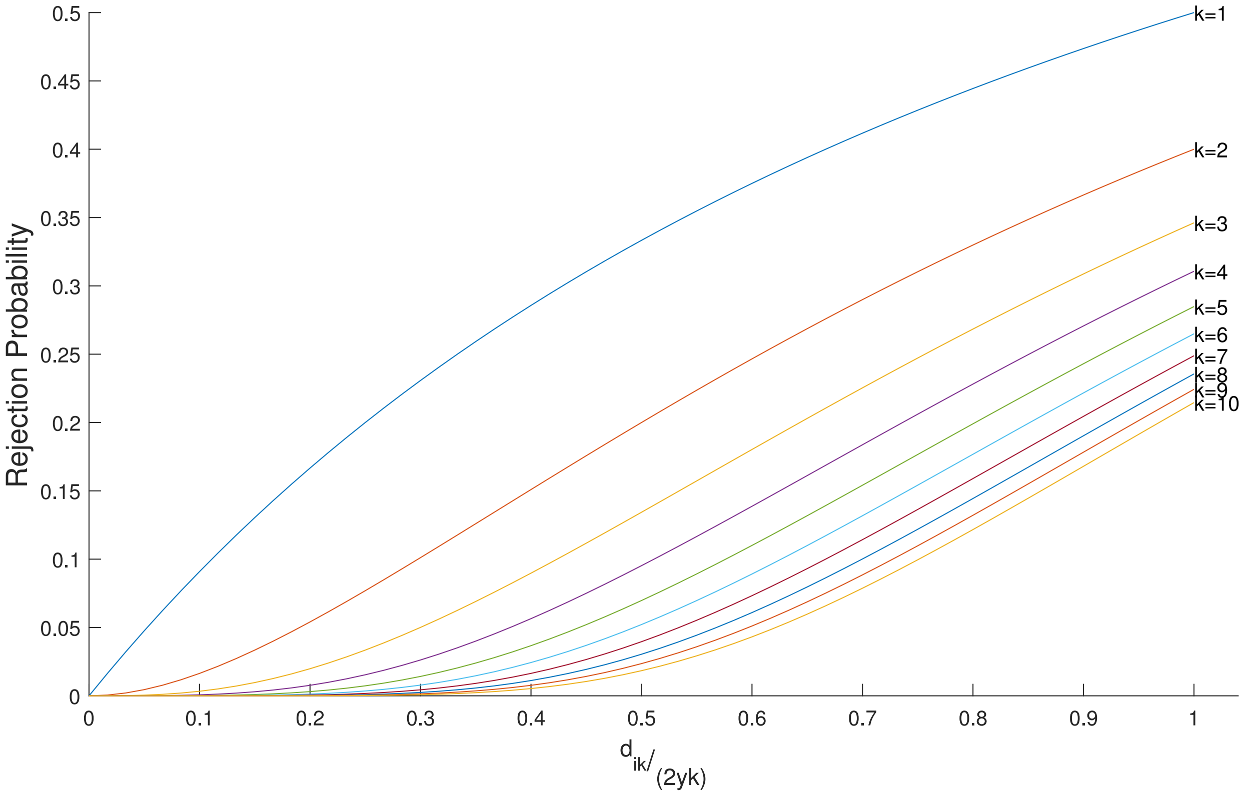


**Fig. 12.** Rejection probability.

car-park of both autonomous and regular vehicles where the regular vehicles are moved with a forklift system. New models are required to capture such interaction.

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## Appendix A

We show that the rejection probability in each island is a function of the demand $d_{ik}$ and the number of spots in island $i$'s stacks ($x_{ik}$). Since a vehicle is rejected when the assigned stack is full, the rejection probability is equal to the probability that a stack of island $i$ with capacity $k$ holds $k$ vehicles:

$$P_{ikk}(d_{ik}) = \frac{(d_{ik}/2y)^k/k!}{\sum_{t=0}^{k}(d_{ik}/2y)^t/t!}$$

Based on Constraint (6) and the fact that $x_{ik}$ is a binary variable, the maximum of $d_{ik}$ is equal to $2yk$. Hence, $d_{ik}/(2yk)$ varies between 0 and 1. Fig. 12 depicts the rejection probability based on $d_{ik}/(2yk)$ for different value of $k$.

As mentioned, the problem with the random allocation is that a vehicle might be rejected at one full stack although another stack has enough room to store it. This leads to under prediction of the allowable demand and over-prediction of the rejection probability. Nevertheless, without the random allocation strategy, the model would be intractable. Hence, it is critical that we make this assumption to solve the problem at a strategic level.

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